CHAPTER 4
COHERENCE THEORY

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4.1 GLOSSARY

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<td>I</td>
<td>intensity [use irradiance or field intensity]</td>
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<td>k</td>
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<td>p</td>
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4.2 INTRODUCTION

Classical Coherence Theory

All light sources produce fields that vary in time with highly complicated and irregular waveforms. Because of diffraction, these waveforms are greatly modified as the fields propagate. All light detectors measure the intensity time averaged over the waveform. This measurement depends on the integration time of the detector and the waveform of the light at the detector. Generally this waveform is not precisely known. Classical coherence theory 1–8 is a mathematical model which is very successful in describing the effects of this unknown waveform on the observed measurement of time-averaged intensity. It is based on the electromagnetic wave theory of light as formulated from Maxwell’s equations, and uses statistical techniques to analyze the effects due to fluctuations in the waveform of the field in both time and space.

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Quantum Coherence Theory

Classical coherence theory can deal very well with almost all presently known optical coherence phenomena; however, a few laboratory experiments require a much more complicated mathematical model, quantum coherence theory, to explain them. This theory is not based on classical statistical theory, but is based on quantum electrodynamics. While the mathematical model underlying classical coherence theory uses simple calculus, quantum coherence theory uses the Hilbert space formulation of abstract linear algebra, which is very awkward to apply to most engineering problems. Fortunately, quantum coherence theory appears to be essential as a model only for certain unusual (even though scientifically very interesting) phenomena such as squeezed light states and photon antibunching. All observed naturally occurring phenomena outside of the laboratory appear to be modeled properly by classical coherence theory or by an approximate semiclassical quantum theory. This article will deal only with the simple classical model.

4.3 SOME ELEMENTARY CLASSICAL CONCEPTS

Analytical Signal Representation

Solutions of the time-dependent, macroscopic Maxwell’s equations yield six scalar components of the electric and the magnetic fields which are functions of both time and position in space. As in conventional diffraction theory, it is much more convenient to treat monochromatic fields than it is to deal with fields that have complicated time dependences. Therefore, each of these scalar components is usually represented at some typical point in space (given with respect to some arbitrary origin by the radius vector \( \mathbf{x} = (x, y, z) \)) by a superposition of monochromatic real scalar components. Thus the field amplitude for a typical monochromatic component of the field with radial frequency \( v \) is given by

\[
U_r(x, v) = U_0(x) \cos \left[ f(x) - ov \right]
\]

where \( U_0(x) \) is the field magnitude and \( f(x) \) is the phase. Trigonometric functions like that in Eq. (1) are awkward to manipulate. This is very well known in electrical circuit theory. Thus, just as in circuit theory, it is conventional to represent this field amplitude by a “phasor” defined by

\[
U(x, \omega) = U_0(x) e^{i f(x)}
\]

The purpose for using this complex field amplitude, just as in circuit theory, is to eliminate the need for trigonometric identities when adding or multiplying field amplitudes. A time-dependent complex analytic signal (viz., Ref. 15, sec. 10.2) is usually defined as the Fourier transform of this phasor, i.e.,

\[
u(x, t) = \int_0^\infty U(x, \omega) e^{-i\omega t} d\omega
\]

The integration in Eq. (3) is only required from zero to infinity because the phasor is defined with hermitian symmetry about the origin, i.e., \( U(-x, \omega) = U^*(x, \omega) \). Therefore, all of the information is contained within the domain from zero to infinity. To obtain the actual field component from the analytical signal just take the real part of it. The Fourier transform in Eq. (3) is well defined if the analytical signal represents a deterministic field. However, if the light is partially coherent, then the analytic signal is usually taken to be a stationary random process. In this case the Fourier inverse of Eq. (3) does not exist. It is then possible to understand the spectral decomposition given by Eq. (3) only within the theory of generalized functions (viz., see Ref. 16, the appendix on generalized functions, and Ref. 17, pp. 25–30).
Scalar Field Amplitude

Each monochromatic component of an arbitrary deterministic light field propagating through a homogeneous, isotropic medium can always be represented using an angular spectrum of plane waves for each of the six scalar components of the vector field. The six angular spectra are coupled together by Maxwell’s equations so that only two are independent. Any two of the six angular spectra can be used to define two scalar fields from which the complete vector field can be determined. A polarized light field can be represented in this way by only one scalar field. Thus it is often possible to represent one polarized component of a vector electromagnetic field by a single scalar field. It has also been found useful to represent completely unpolarized light by a single scalar field. In more complicated cases, where the polarization properties of the light are important, a vector theory is sometimes needed as discussed later under Explicit Vector Representations.

Temporal Coherence and Coherence Time

Within a short enough period of time, the time dependence of any light field at a point in space can be very closely approximated by a sine wave (Ref. 15, sec. 7.5.8). The length of time for which this is a good approximation is usually called the coherence time \( \Delta \tau \). The coherence time is simply related to the spectral bandwidth for any light wave by the uncertainty principle, i.e.,

\[
\Delta \tau \Delta \omega \geq 1 \quad (4)
\]

For a light wave which is also highly directional within some region of space (like a beam) so that it propagates generally in some fixed direction (given by the unit vector \( \mathbf{p} \)), the field amplitude is given by

\[
u(x, t) = f(\mathbf{p} \cdot \mathbf{x} - ct)\quad (5)
\]

Such a traveling wave will be approximately sinusoidal (and hence coherent) over some coherence length \( \Delta \ell \) in the direction of \( \mathbf{p} \) where from Eq. (4) we see that

\[
\Delta \ell = c \Delta \tau = c / \Delta \omega \quad (6)
\]

so that the coherence length varies inversely with bandwidth.

Spatial Coherence and Coherence Area

The time-dependent waveform for any light field is approximately the same at any point within a sufficiently small volume of space called the coherence volume. The projection of this volume onto a surface is termed a coherence area. If we have a field that, within some region, is roughly directional so that its field amplitude is given by Eq. (5), then the coherence length gives the dimension of the coherence volume in the direction of propagation \( \mathbf{p} \), and the coherence area gives the dimensions of the coherence volume normal to this direction.

Measurements of Coherence

Coherence is usually measured by some form of interferometer that takes light from two test points in a light field, \( \mathbf{x}_1 \) and \( \mathbf{x}_2 \), and then allows them to interfere after introducing a time advance \( \tau \) in the light from \( \mathbf{x}_1 \) relative to that from \( \mathbf{x}_2 \). If the field intensity of the interference pattern is measured as a function of \( \tau \), then in general it has the form (see Ref. 15, sec. 10.3.1)

\[
I(\tau) = I(\mathbf{x}_1) + I(\mathbf{x}_2) + 2 \text{ Real } (\Gamma_{12}(\tau)) \quad (7)
\]
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where \( I(x_i) \) is the intensity at the \( i \)th test point, and \( \Gamma_{cc}(\tau) \) is the mutual coherence function which measures the \( \tau \) advanced correlation between the waveforms at the two test points (as subsequently defined under “Mutual Coherence Function”). There are many interferometers which have been developed to measure \( \Gamma_{cc}(\tau) \) in this way. One of the earliest techniques was developed by Thompson and Wolf.\(^{22}\) They used a diffractometer to measure the coherence over a surface normal to the direction of propagation for a collimated beam from a partially coherent source. More recently, Carter\(^{23}\) used an interferometer made from a grating and a microscope to similarly measure the coherence of a beam transverse to the direction of propagation.

4.4 DEFINITIONS OF COHERENCE FUNCTIONS

Mutual Coherence Function

In an older form of coherence theory\(^1\) the principal coherence function was the mutual coherence function defined by\(^{24}\)

\[
\Gamma_{cc}(\tau) = \frac{1}{T} \int_{-\tau}^{\tau} u(x_1, t + \tau) u^*(x_2, t) \, dt
\] (8)

where \( u(x, t) \) represents the complex analytic time-dependent signal at some point \( x \) and some time \( t \) as defined in Eq. (3). This definition was originally motivated by the fact that the intensity, as actually measured, is precisely this time averaged function with \( x_1 = x_2 \) and \( \tau = 0 \), and that this function is the most readily measured since it appears directly in Eq. (7). Thus it was clearly possible to measure \( \Gamma_{cc}(\tau) \) over some input plane, propagate it to an output plane,\(^{25}\) and then find the intensity over the output plane from \( \Gamma_{cc}(\tau) \). It was assumed in this definition in Eq. (8) that \( u(x, t) \) is stationary in time so that \( \Gamma_{cc}(\tau) \) is only a function of \( \tau \) and not of \( t \). In most of the older literature, sharp brackets were usually used to represent this time average rather than an ensemble average (see Ref. 15, sec. 10.3.1). In the early 1960s it was found to be much more convenient to treat \( u(x, t) \) as an ergodic, stationary random process so that Eq. (8) could be replaced by

\[
\Gamma_{cc}(\tau) = \langle u(x_1, t + \tau) u^*(x_2, t) \rangle
\] (9)

where (everywhere in this article) the sharp brackets denote an ensemble average. After the change to ensemble averages the cross-spectral density function (to be defined shortly) became the most used correlation function in the coherence literature, because of the simpler and more general rules for its propagation (as discussed later under “Two Representations”).

Complex Degree of Coherence

To obtain a function that depends only on the coherence properties of a light field it is often useful to normalize the mutual coherence function in the manner of

\[
\gamma_{cc}(\tau) = \frac{\langle u(x_1, t + \tau) u^*(x_2, t) \rangle}{\sqrt{\langle u(x_1, t) u^*(x_1, t) \rangle \langle u(x_2, t) u^*(x_2, t) \rangle}}
\] (10)

This is called the complex degree of coherence. It is a properly normalized correlation coefficient, so that \( \gamma_{cc}(0) = 1 \). This indicates that the field at a point in space must always be perfectly coherent with itself. All other values of \( \gamma_{cc}(\tau) \) are generally complex with an amplitude less than one. This indicates that the fields at two different points, or at the same point after a time delay \( \tau \), are generally less than perfectly coherent with each
other. The magnitude of the complete degree of spatial coherence (from zero to one) is a measure of the mutual coherence between the fields at the two test points and after a time delay $\tau$.

**Cross-Spectral Density Function**

Just as in classical diffraction theory, it is much easier to propagate monochromatic light than light with a complicated time waveform. Thus the most frequently used coherence function is the cross-spectral density function, $W_\omega(x_1, x_2)$, which is the ensemble-averaged correlation function between a typical monochromatic component of the field at some point $x_1$ with the complex conjugate of the same component of the field at some other point $x_2$. It may be defined by

$$
\delta(\omega - \omega')W_\omega(x_1, x_2) = \langle U(x_1, \omega)U^*(x_2, \omega') \rangle
$$

(11)

The amplitude $U(x, \omega)$ for a field of arbitrary coherence is taken to be random variable. Thus $U(x, \omega)$ represents an ensemble of all of the possible fields, each of which is represented by a complex phasor amplitude like that defined in Eq. (2). The sharp brackets denote an ensemble average over all of these possible fields weighted by the probability for each of them to occur. The correlation functions defined by Eqs. (11) and (9) are related by the Fourier transform pairs

$$
\Gamma_{12}(\tau) = \int_0^\infty W_\omega(x_1, x_2)e^{-i\omega\tau}d\omega
$$

(12)

and

$$
W_\omega(x_1, x_2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Gamma_{12}(\tau)e^{i\omega\tau}d\tau
$$

(13)

which is easily shown, formally, by substitution from Eq. (3) into (9) and then using (11). These relations represent a form of the generalized Wiener-Khintchine theorem (see Ref. 26, pp. 107–108).

**Complex Degree of Spectral Coherence**

Because the cross-spectral density function contains information about both the intensity (see “Intensity,” which follows shortly) and the coherence of the field, it is useful to define another coherence function which describes the coherence properties only. This is the complex degree of spectral coherence (not to be confused with the complex degree of spatial coherence, which is a totally different function), which is usually defined by

$$
\mu_\omega(x_1, x_2) = \frac{W_\omega(x_1, x_2)}{\sqrt{W_\omega(x_1, x_1)W_\omega(x_2, x_2)}}
$$

(14)

It is easy to show that this function is a properly normalized correlation coefficient which is always equal to unity if the field points are brought together, and is always less than or equal to unity as they are separated. If the magnitude of this function is less than unity it indicates less-than-perfect coherence. If the magnitude is zero it indicates complete incoherence between the field amplitudes at the two test points. For most partially coherent fields the cross-spectral density function has significantly large values only for point separations which keep the two field points within the same coherence volume. This function depends only on the positions of the points and the single radial frequency that the field
components at the two points share. Field components of different frequency are always uncorrelated (and therefore incoherent), even at the same point.

**Spectrum and Normalized Spectrum**

Recently, the changes in the spectrum of light due to propagation have been studied using coherence theory. It is therefore useful to define the spectrum of light as just the monochromatic intensity (which is just the trace of the cross-spectral density function) as a function of omega, and the spectrum of a primary source as a very similar function, i.e.,

\[ S_p(x, \omega) = \langle U_p(x)U_p^*(x) \rangle = W_p(x, x) \]

\[ S_U(x, \omega) = \langle \rho_p(x)\rho_p(x) \rangle = W_U(x, x) \]

where the subscript \(Q\) indicates that this is a primary source spectrum, and the subscript \(U\) indicates that this is a field spectrum. The spectrum for the primary source is a function of the phasor \(r_p(x)\) which represents the currents and charges in this source as discussed under "Primary Sources" in the next section. It is also useful to normalize these spectra in the manner

\[ s_A(x, \omega) = \frac{S_A(x, \omega)}{\int_0^\infty S_A(x, \omega) \, d\omega} \]

where the subscript \(A\) can indicate either \(U\) or \(Q\), and the normalized spectrum has the property

\[ \int_0^\infty s_A(x, \omega) \, d\omega = 1 \]

so that it is independent of the total intensity.

**Angular Correlation Function**

A new coherence function, introduced for use with the angular spectrum expansion of a monochromatic component of the field,\(^{28}\) is the angular correlation function defined by

\[ \mathcal{A}_w(p_1, p_2) = \langle A_w(p_1), A_w^*(p_2) \rangle \]

where \(A_w(p)\) is the angular spectrum which gives the complex amplitude of the plane wave component of the field which propagates in the direction given by the unit vector \(p\). This is related to the cross-spectral density function over the \(z = 0\) plane by the equation

\[ \mathcal{A}(p_1, p_2) = \frac{1}{A^2} \int \int \int \int W_w^{(0)}(x_j, x_k) e^{-i(p_1 \cdot x_j - p_2 \cdot x_k)} \, dx_j \, dx_k \]

This is the four-dimensional Fourier transform of the cross-spectral density function over the \(z = 0\) plane. It represents a correlation function between the complex amplitudes of two plane wave components of the field propagating in the directions given by the unit vectors \(p_1\) and \(p_2\), respectively. It can be used to calculate the cross-spectral density function (as described later under "Angular Spectrum Representation") between any pair of points away from the \(z = 0\) plane, assuming that the field propagates in a source-free homogeneous medium. In this article we will use single-primed vectors, as in Eq. (19), to represent radius vectors from the origin to points within the \(z = 0\) plane, i.e., \(x' = (x', y', 0)\), as shown in Fig. 1.
FIGURE 1 Illustrating the coordinate system and notation used for planar sources in the \( z = 0 \) plane.

All other vectors, such as \( \mathbf{x} \) or \( \mathbf{x}' \), are to be taken as three-dimensional vectors. Generally \( s \) and \( p \) are three-dimensional unit vectors indicating directions from the origin, a superscript \((0)\) on a function indicates that it is the boundary condition for that function over the \( z = 0 \) plane, and a superscript \((\infty)\) on a function indicates that it is the asymptotic value for that function on a sphere of constant radius \( R \) from the origin as \( R \to \infty \).

Intensity

The intensity is usually considered to be the observable quantity in coherence theory. Originally it was defined to be the trace of the mutual coherence function as defined by Eq. (8), i.e.,

\[
I(x_1) \equiv \Gamma_{11}(0) = \frac{1}{T-T} \int_{-T}^{T} u(x_1, t) u^*(x_1, t) \, dt
\]  

(20)

which is always real. Thus it is the time-averaged square magnitude of the analytic signal. This represents the measurement obtained by the electromagnetic power detectors always used to detect light fields. Since the change to ensemble averages in coherence theory, it is almost always assumed that the analytic signal is an ergodic random process so that the intensity can be obtained from the equation

\[
I(x_1) \equiv \Gamma_{11}(0) = \langle u(x_1, t) u^*(x_1, t) \rangle
\]  

(21)

where the sharp brackets indicate an ensemble average. Usually, in most recent coherence-theory papers, the intensity calculated is actually the spectrum, which is equivalent to the intensity of a single monochromatic component of the field which is defined as the trace of the cross-spectral density function as given by

\[
L_{\nu}(x_1) \equiv W_{\nu}(x_1, x_1) = \langle U(x_1, \omega) U^*(x_1, \omega) \rangle
\]  

(22)

Since different monochromatic components of the field are mutually incoherent and cannot interfere, we can always calculate the intensity of the total field as the sum over the intensities of its monochromatic components in the manner

\[
I(x_1) \equiv \int_{0}^{\infty} L_{\nu}(x_1) \, d\omega
\]  

(23)

Since in most papers on coherence theory the subscript \( \omega \) is usually dropped, the reader should be careful to observe whether or not the intensity calculated is for a monochromatic component of the field only. If it is, the total measurable intensity can be obtained simply by summing over all \( \omega \), as indicated in Eq. (23).


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Radiant Emittance

In classical radiometry the radiant emittance is defined to be the power radiated into the far field by a planar source per unit area. A wave function with some of the properties of radiant emittance has been defined by Marchand and Wolf using coherence theory (see Ref. 29, eq. (32), and Ref. 30). However, because of interference effects, the far-field energy cannot be subdivided into components that can be traced back to the area of the source that produced them. The result is that the radiant emittance defined by Marchand and Wolf is not nonnegative definite (as it is in classical radiometry) except for the special case of a completely incoherent source. Since, as discussed in the next section under “Perfectly Incoherent Source,” perfectly incoherent sources exist only as a limiting case, radiant emittance has not been found to be a very useful concept in coherence theory.

Radiant Intensity

In classical radiometry the radiant intensity is defined to be the power radiated from a planar source into a unit solid angle with respect to an origin at the center of the surface. This can be interpreted in coherence theory as the field intensity over a portion of a surface in the far field of the source, in some direction given by the unit vector \( \mathbf{s} \), which subtends a unit solid angle from the source. Thus the radiant intensity for a monochromatic component of the field can be defined in coherence theory as

\[
J_v(s) = \lim_{R \to \infty} W(\omega, R \mathbf{s}) R^2 \quad (24)
\]

To obtain the total radiant intensity we need only sum this function over all \( \omega \).

Radiance

In classical coherence theory the radiance function is the power radiated from a unit area on a planar source into a unit solid angle with respect to an origin at the center of the source. In the geometrical optics model, from which this concept originally came, it is consistent to talk about particles of light leaving an area at a specified point on a surface to travel in some specified direction. However, in a wave theory, wave position and wave direction are Fourier conjugate variables. We can have a configuration space wave function (position) or a momentum space wave function (direction), but not a wave function that depends independently on both position and direction. Thus the behavior of a wave does not necessarily conform to a model which utilizes a radiance function. Most naturally occurring sources are quasi-homogeneous (discussed later). For such sources, a radiance function for a typical monochromatic component of the field can be defined as the Wigner distribution function of the cross-spectral density function over the \( z = 0 \) plane, which is given by

\[
B_v(x', s) = \frac{\cos \theta}{k^2} \int \int W(\omega'(x' + x'./2, x'' - x'./2)e^{-ikx'.} d^2x'. \quad (25)
\]

where \( \theta \) is the angle that the unit vector \( s \) makes with the \( +z \) axis. For quasi-homogeneous fields this can be associated with the energy radiated from some point \( x' \) into the far field in the direction \( s \). Such a definition for radiance also works approximately for some other light fields, but for light which does not come from a quasi-homogeneous source, no such definition is either completely equivalent to a classical radiance function or unique as an approximation to it. Much progress has been made toward representing a more general class of fields using a radiance function. In general, waves do not have radiance functions.
Higher-Order Coherence Functions

In general, the statistical properties of a random variable are uniquely defined by the probability density function which can be expanded into a series which contains correlation functions of all orders. Thus, in general, all orders of correlation functions are necessary to completely define the statistical properties of the field. In classical coherence theory we usually assume that the partially coherent fields arise from many independent sources so that, by the central limit theorem of statistics, the probability distribution function for the real and imaginary components of the phasor field amplitude are zero-mean gaussian random variables (See Ref. 8, sec. 2.72e). Thus, from the gaussian moment theorem, the field correlation functions of any order can be calculated from the second-order correlation functions, for example,

$$\langle U(x_1, \omega)U^*(x_2, \omega)\rangle = I_\omega(x_1)I_\omega(x_2) + |W_\omega(x_1, x_2)|^2$$  (26)

Thus, for gaussian fields, the second-order correlation functions used in coherence theory completely define the statistical properties of the field. Some experiments, such as those involving intensity interferometry, actually measure fourth- or higher-order correlation functions.\textsuperscript{37}

4.5 MODEL SOURCES

Primary Sources

In coherence theory it is useful to talk about primary and secondary sources. A primary source distribution is the usual source represented by the actual charge and current distribution which give rise to the field. For propagation from a primary source through a source-free media, the field amplitude is defined by the inhomogeneous Helmholtz equation [see Ref. 38, eq. (6.37)], i.e.,

$$\left(\nabla^2 + \frac{\omega^2}{c^2}\right)U_\omega(x) = -4\pi\rho_\omega(x)$$  (27)

where $\rho_\omega(x)$ represents the charge-current distribution in the usual manner. A solution to this wave equation gives

$$W_\omega(x_1, x_2) = \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int W_\omega(x_1', x_2')K(x_1, x_1')K^*(x_2, x_2')d^3x_1'd^3x_2'$$  (28)

for the cross-spectral density function, where $k = \omega/c = 2\pi/\lambda$

$$K(x, x') = \frac{e^{ik|x-x'|}}{|x-x'|}$$  (29)

is the free-space propagator for a primary source, $W_\omega(x_1, x_2)$ is the cross-spectral density function for the fields (with suppressed $\omega$-dependence), as defined by Eq. (11), and

$$W_\omega(x_1', x_2') = \langle \rho_\omega(x_1')\rho_\omega^*(x_2') \rangle$$  (30)

is the cross-spectral density function for the source. This three-dimensional primary source
can be easily reduced to a two-dimensional source over the $z = 0$ plane by simply defining the charge current distribution to be

$$\rho_\omega(x) = \rho_\omega(x, y) \delta(z)$$

(31)

where $\delta(z)$ is the Dirac delta function.

Secondary Sources

Often, however, it is more convenient to consider a field which arises from sources outside of the region in space in which the fields are of interest. Then it is sometime useful to work with boundary conditions for the field amplitude over some surface bounding the region of interest. In many coherence problems these boundary conditions are called a planar secondary source even though they are actually treated as boundary conditions. For example, most conventional diffraction equations assume the boundary condition over the $z = 0$ plane is known and use it to calculate the fields in the $z > 0$ half-space. Then the field obeys the homogeneous Helmholtz equation [See Ref. 38, eq. (7.8)], i.e.,

$$\left( \nabla^2 + \frac{\omega^2}{c^2} \right) U(x) = 0$$

(32)

which has the solution

$$W(x_i, x_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_{\omega}^{(0)}(x_1, x_1') h(x_1, x_1') h^*(x_2, x_2') \, d^2 x_1 \, d^2 x_1'$$

(33)

where $W_{\omega}^{(0)}(x_1, x_1')$ is the boundary condition for the cross-spectral density function of the fields over the $z = 0$ plane (as shown in Fig. 1), $W_{\omega}(x_1, x_2)$ is the cross-spectral density function anywhere in the $z > 0$ half-space, and

$$h(x, x') = -\frac{1}{2\pi} \frac{e^{ik|x-x'|}}{dz |x-x'|}$$

(34)

is the free-space propagator for the field amplitude. This is a common example of a secondary source over the $z = 0$ plane.

Perfectly Coherent Source

The definition of a perfectly coherent source, within the theory of partial coherence, is somewhat complicated. The light from an ideal monochromatic source is, of course, always perfectly coherent. Such a field produces high-contrast interference patterns when its intensity is detected by any suitable instrument. However, it is possible that light fields exist that are not monochromatic but can also produce similar interference patterns and therefore must be considered coherent. The ability of light fields to produce interference patterns at a detector is measured most directly by the complex degree of coherence.
\(\gamma_{12}(\tau),\) defined by Eq. (10). If a field has a complex degree of coherence that has unit magnitude for every value of \(\tau\) and for every point pair throughout some domain \(D\), then light from all points in \(D\) will combine to produce high-contrast interference fringes.\(^{39}\) Such a field is defined to be perfectly coherent within \(D\). Mandel and Wolf\(^{27}\) have shown that the mutual coherence function for such a field factors within \(D\) in the manner

\[\Gamma_{12}(\tau) = \psi(x_1)\psi^*(x_2)e^{-i\omega\tau}\]  

(35)

We will take Eq. (35) to be the definition of a perfectly coherent field. Coherence is not as easily defined in the space-frequency domain because it depends on the spectrum of the light as well as on the complex degree of spectral coherence. For example, consider a field for which every monochromatic component is characterized by a complex degree of spectral coherence which has unit magnitude between all point pairs within some domain \(D\). Mandel and Wolf\(^{40}\) have shown that for such a field the cross-spectral density function within \(D\) factors is

\[W_{\omega}(\mathbf{x}_1, \mathbf{x}_2) = \tilde{U}(\mathbf{x}_1, \omega)\tilde{U}^*(\mathbf{x}_2, \omega)\]  

(36)

However, even if Eq. (36) holds for this field within \(D\), the field may not be perfectly coherent [as perfect coherence is defined by Eq. (35)] between all points within the domain. In fact it can be completely incoherent between some points within \(D\).\(^{41}\) A secondary source covering the \(z = 0\) plane with a cross-spectral density function over that plane which factors as given by Eq. (36) will produce a field in the \(z > 0\) half-space (filled with free space or a homogeneous, isotropic dielectric) which has a cross-spectral density function that factors in the same manner everywhere in the half-space. This can be easily shown by substitution from Eq. (36) into Eq. (28), using Eq. (29). But, even if this is true for every monochromatic component of the field, Eq. (35) may not hold within the half-space for every point pair, so we cannot say that the field is perfectly coherent there. Perfectly coherent light sources never actually occur, but sometimes the light from a laser can behave approximately in this way over some coherence volume which is usefully large. Radio waves often behave in this manner over very large coherence volumes.

**Quasi-Monochromatic Source**

In many problems it is more useful not to assume that a field is strictly monochromatic but instead to assume that it is only quasi-monochromatic so that the time-dependent field amplitude can be approximated by

\[u(x, t) = u_0(x, t)e^{-i\omega t}\]  

(37)

where \(u_0(x, t)\) is a random process which varies much more slowly in time than \(e^{-i\omega t}\). Then the mutual coherence function and the complex degree of spatial coherence can be usefully approximated by (see Ref. 15, sec. 10.4.1)

\[\Gamma_{12}(\tau) = \Gamma_{12}(0)e^{-i\omega\tau}\]

\[\gamma_{12}(\tau) = \gamma_{12}(0)e^{-i\omega\tau}\]  

(38)

within coherence times much less than the reciprocal bandwidth of the field, i.e., \(\Delta\tau \ll 1/\Delta\omega\). In the pre-1960 coherence literature, \(\Gamma_{12}(0)\) was called the *mutual intensity*. Monochromatic diffraction theory was then used to define the propagation properties of this monochromatic function. It was used instead of the cross-spectral density function to formulate the theory for the propagation of a partially coherent quasi-monochromatic field. While this earlier form of the theory was limited by the quasi-monochromatic approximation and was therefore not appropriate for very wideband light, the newer
formulation (in terms of the cross-spectral density function) makes no assumptions about
the spectrum of the light and can be applied generally. The quasi-monochromatic
approximation is still very useful when dealing with radiation of very high coherence.42

Schell Model Source

Other, more general, source models have been developed. The most general of these is the
Schell model source (see Ref. 43, sec. 7.5, and Ref. 44), for which we only assume that the
complex degree of spectral coherence for either a primary or secondary source is
stationary in space, so that from Eq. (14) we have

$$W_A(x_1, x_2) = \mu_A(x_1 - x_2) \sqrt{W_A(x_1, x_1) W_A(x_2, x_2)}$$ (39)

where the subscript $A$ stands for $U$ in the case of a Schell model secondary source and $Q$
in the case of a Schell model primary source. The Schell model does not assume low
coherence and, therefore, can be applied to spatially stationary light fields of any state of
coherence. The Schell model of the form shown in Eq. (39) has been used to represent
both three-dimensional primary sources45,46 and two-dimensional secondary sources.43,47,48

Quasi-Homogeneous Source

If the intensity of a Schell model source is essentially constant over any coherence area,
then Eq. (39) may be approximated by

$$W_A(x_1, x_2) = \mu_A(x_1 - x_2) I_A([x_1 + x_2]/2)$$ (40)

where the subscript $A$ can be either $U$, for the case of a quasi-homogeneous secondary
source or $Q$, for the case of a quasi-homogeneous primary source. This equation is very
useful in coherence theory because of the important exact mathematical identity.49

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mu_\omega^{(0)}(x'_1 - x'_2) I^{(0)}_w([x'_1 + x'_2]/2) e^{-i k (x'_1 x'_2)} dx'_1 dy'_1 dx'_2 dy'_2$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mu_\omega^{(0)}(x'_1) e^{-i k (x'_1 x'_1)} dx'_1 dy'_1 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} I^{(0)}_w(x'_2) e^{-i k (x'_2 x'_2)} dx'_2 dy'_2$$ (41)

where

$$x'_1 = (x'_1 + x'_2)/2 \quad x'_2 = x'_1 - x'_2$$

and

$$p'_1 = (p_1 + p_2)/2 \quad p'_2 = p_1 - p_2$$

which allows the four-dimensional Fourier transforms that occur in propagating the
correlation functions for secondary sources [for example, Eqs. (49) or (54)] to be factored
into a product of two-dimensional Fourier transforms. An equivalent identity also holds
for the six-dimensional Fourier transform of the cross-spectral density function for a
primary source, reducing it to the product of two three-dimensional Fourier transforms.
This is equally useful in dealing with propagation from primary sources [for example, Eq.
(53)]. This model is very good for representing two-dimensional secondary sources with
sufficiently low coherence that the intensity does not vary over the coherence area on the
input plane.49,50 It has also been applied to primary three-dimensional sources.45,46 to
primary and secondary two-dimensional sources, and to three-dimensional scattering potentials.

Perfectly Incoherent Source

If the coherence volume of a field becomes much smaller than any other dimensions of interest in the problem, then the field is said to be incoherent. It is believed that no field can be incoherent over dimensions smaller than the order of a light wavelength. An incoherent field can be taken as a special case of a quasi-homogeneous field for which the complex degree of spectral coherence is approximated by a Dirac delta function, i.e.,

\[ W_A(x_1, x_2) = I(x_1) \delta(x_1 - x_2) \]

\[ W_Q(x_1, x_2) = I(x_1) \delta(x_1 - x_2) \]

where the two-dimensional Dirac delta function is used for any two-dimensional source and a three-dimensional Dirac delta function is used only for a three-dimensional primary source. Even though this approximation is widely used, it is not a good representation for the thermal sources that predominate in nature. For example, the radiant intensity from a planar, incoherent source is not in agreement with Lambert’s law. For thermal sources the following model is much better.

Thermal (Lambertian) Source

For a planar, quasi-homogeneous source to have a radiant intensity in agreement with Lambert’s law it is necessary for the complex degree of spectral coherence to have the form

\[ \mu_{a}(x_1 - x_2) = \frac{\sin (k |x_1 - x_2|)}{k |x_1 - x_2|} \]

(43)

where arbitrary spatial frequency components with periods less than a wavelength can be added since they do not affect the far field. It can also be shown that, under frequently obtained conditions, blackbody radiation has such a complex degree of spectral coherence. It is believed that most naturally occurring light can be modeled as quasi-homogeneous with this correlation function.

4.6 PROPAGATION

Perfectly Coherent Light

Perfectly coherent light propagates according to conventional diffraction theory. By substitution from Eq. (36) into Eq. (33) using Eq. (34) we obtain

\[ U_\omega(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} U^{(0)}(x') \frac{1}{2\pi dz'} \frac{e^{i\omega \cdot z}}{d^2z'} dx' \]

(44)

which is just Rayleigh’s diffraction integral of the first kind. Perfectly coherent light does
not lose coherence as it propagates through any medium which is time-independent. For perfectly coherent light propagating in time-independent media, coherence theory is not needed.

**Hopkin’s Formula**

In 1951 Hopkins published a formula for the complex degree of spatial coherence for the field from a planar, secondary, incoherent, quasi-monochromatic source after propagating through a linear optical system with spread function \( h(x, x') \), i.e.,

\[
\gamma_{12}(0) = \frac{1}{\sqrt{I_I(x_1)I_I(x_2)}} \int \int I_1^0(x)h(x_1 - x)h^{*}(x_2 - x) \, d^2x
\]  

(45)

where \( I_1^0(x) \) is the intensity over the source plane. This formula can be greatly generalized to give the complex degree of spectral coherence for the field from any planar, quasi-homogeneous, secondary source after transmission through this linear optical system, i.e.,

\[
\mu_{12}(x_1, x_2) = \frac{1}{\sqrt{I_I(x_1)I_I(x_2)}} \int \int I_0^0(x')h(x_1, x')h^{*}(x_2, x') \, d^2x'
\]  

(46)

provided that the spread function \( h(x, x') \) can be assumed to be constant in its \( x' \) dependence over any coherence area in the source plane.

**van Cittert–Zernike Theorem**

Hopkins’ formula can be specialized for free-space propagation to calculate the far-field coherence properties of planar, secondary, quasi-homogeneous sources of low coherence. In 1934 van Cittert and, later, Zernike derived a formula equivalent to

\[
\mu_{12}(x_1, x_2) = \frac{1}{\sqrt{I_I(x_1)I_I(x_2)}} \int \int I_0^0(x')e^{ik\left(x_1 - x_2\right)/R} \, d^2x'
\]  

(47)

for the complex degree of spectral coherence between any pair of points in the field radiated from an incoherent planar source, assuming that the points are not within a few wavelengths of the source. We can obtain Eq. (47) by substitution from Eq. (34) into Eq. (46) and then approximating the propagator in a standard manner [see Ref. 61, eq. (7)]. Assume next, that the source area is contained within a circle of radius \( a \) about the origin in the source plane as shown in Fig. 4. Then, if the field points are both located on a sphere of radius \( R \), which is outside of the Rayleigh range of the origin (i.e., \( |x| = R \gg ka^2 \)), we can apply the Fraunhofer approximation to Eq. (47) to obtain

\[
\mu_{12}(x_1, x_2) = \frac{1}{\sqrt{I_I(x_1)I_I(x_2)}} \int \int I_0^0(x')e^{ik\left(x_1 - x_2\right)/R} \, d^2x'
\]  

(48)

This formula is very important in coherence theory. It shows that the complex degree of spectral coherence from any planar, incoherent source with an intensity distribution \( I_0^0(x') \) has the same dependence on \( (x_1 - x_2) \), over a sphere of radius \( R \) in the far field, as the diffraction pattern from a closely related perfectly coherent planar source with a real
amplitude distribution proportional to $I_{00}^{(0)}(x')$ (see Ref. 15, sec. 10.4.2a). Equation (48) can also be applied to a planar, quasi-homogeneous source that is not necessarily incoherent, as will be shown later under “Reciprocity Theorem.”

Angular Spectrum Representation

Much more general equations for propagation of the cross-spectral density function can be obtained. Equation (33) is one such expression. Another can be found if we expand the fields in a source-free region of space into an angular spectrum of plane waves. Then we find that the cross-spectral density function over any plane can be calculated from the same function over a parallel plane using a linear systems approach. For an example, consider the two planes illustrated in Fig. 2.

We assume that the cross-spectral density function is known over the $z = 0$ plane in the figure, and we wish to calculate this function over the $z = d$ plane. To do this we first take the Fourier transform of the cross-spectral density function over the $z = 0$ plane according to Eq. (19)

$$A_{10}(p_1, p_2) = \frac{1}{\lambda^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W^{(0)}(x'_1, x'_2) e^{-ik(p_1 x'_1 + p_2 x'_2)} d^2x'_1 d^2x'_2$$

(49)

to obtain the angular correlation function in which all of the phase differences between the plane wave amplitudes are given relative to the point at the origin. Second, we shift the phase reference from the origin to the point on the $z$ axis in the output plane by multiplying the angular correlation function by a transfer function, i.e.,

$$A_{out}(p_1, p_2) = A_{in}(p_1, p_2) \exp \left[ ik(m_1 - m_2) d \right]$$

(50)

where $d$ is the distance from the input to the output plane along the $z$ axis (for back propagation $d$ will be negative) and $m_i$ is the third component of the unit vector $p_i = (p_i, q_i, m_i), i = 1$ or 2, which is defined by

$$m_i = \sqrt{1 - p_i^2 - q_i^2}, \text{ if } p_i^2 + q_i^2 \leq 1$$

$$= i\sqrt{p_i^2 + q_i^2 - 1}, \text{ if } p_i^2 + q_i^2 > 1$$

(51)

FIGURE 2 Illustrating the coordinate system for propagation of the cross-spectral density function using the angular spectrum of plane waves.
and is the cosine of the angle that \( p_i \) makes with the \(+z\) axis for real \( m_i \). Finally, to obtain the cross-spectral density function over the output plane, we simply take the Fourier inverse of \( \mathcal{F}_\text{out}(p_1, p_2) \), i.e.,

\[
W^{(d)}(x_1, x_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathcal{F}_\text{out}(p_1, p_2) e^{i k (p_1 x_1 - p_2 x_2)} \, dp_1 \, dp_2
\]

(52)

where, in this equation only, we use \( x \) to represent a two-dimensional radius vector from the point \((0, 0, d)\) to a field point in the \( z = d \) plane, as shown in Fig. 2. This propagation procedure is similar to the method usually used to propagate the field amplitude in Fourier optics. In coherence theory, it is the cross-spectral density function for a field of any state of coherence that is propagated between arbitrary parallel planes using the linear systems procedure. The only condition for the validity of this procedure is that the volume between the planes must either be empty space or a uniform dielectric medium.

**Radiation Field**

The cross-spectral density function far away from any source, which has finite size, can be calculated using a particularly simple equation. Consider a primary, three-dimensional source located within a sphere of radius \( a \), as shown in Fig. 3. For field points, \( x_1 \) and \( x_2 \) which are much farther from the origin than the Rayleigh range \((|x| \gg k a)\), in any direction, a form of the familiar Fraunhofer approximation can be applied to Eq. (28) to obtain [see Ref. 62, eq. (2.5)]

\[
W^{(d)}(x_1, x_2) = \frac{e^{ik|x_1 - x_2|}}{|x_1||x_2|} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_0(x'_1, x'_2) \times e^{ik|x'_1 - x_1|} e^{ik|x'_2 - x_2|} \, dx'_1 \, dx'_2 \, dx'_3
\]

(53)

Thus the cross-spectral density function of the far field is proportional to the six-dimensional Fourier transform of the cross-spectral density function of its sources. A very

![FIGURE 3 Illustrating the coordinate system used to calculate the cross-spectral density function in the far field of a primary, three-dimensional source.](image)
FIGURE 4 Illustrating the coordinate system for calculating the far field cross-spectral density function for a planar, secondary source distribution in the $z=0$ plane. Single-primed coordinates indicate radius vectors from the origin to points within the $z=0$ plane.

A similar expression can also be found for a two-dimensional, secondary source distribution over the $z=0$ plane, as illustrated in Fig. 4.

If the sources are all restricted to the area within a circle about the origin of radius $a$, then the cross-spectral density function for all points which are outside of the Rayleigh range ($|x| > ka^2$) from the origin and also in the $z > 0$ half-space can be found by using a different form of the Fraunhofer approximation in Eq. (33) [see Ref. 62, eq. (3.3)] to get

$$W_U(x_1, x_2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{z_1}{|x_1|} \frac{z_2}{|x_2|} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_{U0}(x_1', x_2')$$

$$\times e^{-i[k(x_1-x_1') + k(x_2-x_2')] / 2 |x_1'^2 + x_2'^2|} d^2x_1' d^2x_2'$$

(54)

Because of both their relative simplicity (as Fourier transforms) and great utility, Eqs. (53) and (54) are very important in coherence theory for calculating both the radiant intensity and the far-field coherence properties of the radiation field from any source.

Representations

Several equations have been described here for propagating the cross-spectral density function. The two most general are Eq. (33), which uses an expansion of the field into spherical Huygens wavelets, and Eqs. (49), (50), and (52), which use an expansion of the field into an angular spectrum of plane waves. These two formulations are completely equivalent. Neither uses any approximation not also used by the other method. The choice of which to use for a particular calculation can be made completely on a basis of convenience. The far-field approximations given by Eqs. (53) and (54), for example, can be derived from either representation. There are two bridges between the spherical and plane wave representations, one given by Weyl’s integral, i.e.,

$$\frac{e^{i[k(x-x')]}}{|x-x'|} = \frac{i}{\lambda} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{m} e^{i[p(x-x') + q(y-y') + m(z-z')]} dp\, dq$$

(55)
where

\[ m = \sqrt{1 - p^2 - q^2} \text{ if } p^2 + q^2 \leq 1, \]
\[ = i\sqrt{p^2 + q^2 - 1} \text{ if } p^2 + q^2 > 1 \]  

and the other by a related integral

\[ \frac{-1}{2\pi} \int \frac{d\mathbf{z}}{\mathbf{z} \cdot \mathbf{x}} = \pm \frac{1}{\lambda^2} \int_{-\infty}^{\infty} e^{ik(p(x - x') + q(y - y') + m(z - z'))} dp dq \]  

which can be easily derived from Eq. (55). In Eq. (57) the \( \pm \) sign holds according to whether \( (z - z') \geq 0 \). With these two equations it is possible to transform back and forth between these two representations.

**Reciprocity Theorem**

The radiation pattern and the complex degree of spectral coherence obey a very useful reciprocity theorem for a quasi-homogeneous source. By substituting from Eq. (40) into Eq. (53), and using Eq. (24) and the six-dimensional form of Eq. (41), it has been shown that the radiant intensity in the direction of the unit vector \( \mathbf{s} \) from any bounded, three-dimensional, quasi-homogeneous primary source distribution is given by [see Ref. 45, eq. (3.11)]

\[ J_v(s) = J_0 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mu_{\mathbf{q}}(\mathbf{x}') e^{-ik\mathbf{x}' \cdot \mathbf{s}} d^3\mathbf{x}' \]  

where \( \mu_{\mathbf{q}}(\mathbf{x}') \) is the (spatially stationary) complex degree of spectral coherence for the source distribution as defined in Eq. (40), and

\[ J_0 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} I_{\mathbf{q}}(\mathbf{x}') d^3\mathbf{x}' \]  

where \( I_{\mathbf{q}}(\mathbf{x}') \) is the intensity defined in Eq. (40). Note that the far-field radiation pattern depends, not on the source intensity distribution, but only on the source coherence. We also find from this calculation that the complete degree of spectral coherence between any two points in the far field of this source is given by [see Ref. 45, eq. (3.15)]

\[ \mu^{(x)}(R_1, s_1, R_2, s_2) = \frac{-e^{ik(R_1 - R_2)}}{J_0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} I_{\mathbf{q}}(\mathbf{x}') e^{ikx' \cdot (s_2 - s_1)} d^3\mathbf{x}' \]  

Note that the coherence of the far field depends, not on the source coherence, but rather on the source intensity distribution, \( I_{\mathbf{q}}(\mathbf{x}') \). Equation (60) is a generalization of the van Cittert–Zernike theorem to three-dimensional, primary quasi-homogeneous sources, which are not necessarily incoherent. Equation (59) is a new theorem, reciprocal to the van Cittert–Zernike theorem, which was discovered by Carter and Wolf.45 Equations (58) and (60), taken together, give a reciprocity relation. For quasi-homogeneous sources, far-field coherence is determined by source intensity alone and the far-field intensity pattern is determined by source coherence alone. Therefore, coherence and intensity are reciprocal when going from the source into the far field. Since most sources which occur naturally are believed to be quasi-homogeneous, this is a very useful theorem. This reciprocity theorem has been found to hold much more generally than just for
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three-dimensional primary sources. For a planar, secondary source, by substitution from Eq. (40) into Eq. (54) and then using Eq. (41), we obtain

$$J_v(s) = J_0 \cos^2 \theta \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mu_0^{(0)}(x',) e^{-ikx' \cdot s} d^2x'. \quad (61)$$

where $\theta$ is the angle that $s$ makes with the $+z$ axis, and where

$$J_0 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} I_0^{(0)}(x',) d^2x'. \quad (62)$$

and we also obtain

$$\mu_0^{(0)}(R_1 s_1, R_2 s_2) = \frac{1}{J_0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} I_0^{(0)}(x',) e^{-ik x' \cdot (s_1 - s_2)} d^2x'. \quad (63)$$

Very similar reciprocity relations also hold for scattering of initially coherent light from quasi-homogeneous scattering potentials, and for the scattering of laser beams from quasi-homogeneous sea waves. Reciprocity relations which apply to fields that are not necessarily quasi-homogeneous have also been obtained.

Nonradiating Sources

One additional important comment must be made about Eq. (58). The integrals appearing in this equation form a three-dimensional Fourier transform of $\mu_0(x')$. However, $J_v(s)$, the radiant intensity that this source radiates into the far field at radial frequency $v$, is a function of only the direction of $s$, which is a unit vector with a constant amplitude equal to one. It then follows that only the values of this transform over a spherical surface of unit radius from the origin affect the far field. Therefore, sources which have a complex degree of spectral coherence, $\mu_0(x')$, which do not have spatial frequencies falling on this sphere, do not radiate at frequency $v$. It appears possible from this fact to have sources in such a state of coherence that they do not radiate at all. Similar comments can also be made in regard to Eq. (53) which apply to all sources, even those that are not quasi-homogeneous. From Eq. (53) it is clear that only sources which have a cross-spectral density function with spatial frequencies (with respect to both $x_1^1$ and $x_2^0$) on a unit sphere can radiate. It is believed that this is closely related to the phase-matching condition in nonlinear optics.

Perfectly Incoherent Sources

For a completely incoherent primary source the far-field radiant intensity, by substitution from Eq. (42) into Eq. (58), is seen to be the same in all directions, independent of the shape of the source. By substitution from Eq. (40) into Eq. (28), and then using Eqs. (14), (29), and (42), we find that the complex degree of spectral coherence for this incoherent source is given by

$$\mu_0(x_1, x_2) = \frac{1}{\sqrt{I(x_1)(x_2)}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} I_0(x') \frac{e^{ik|\mathbf{x}_1 - \mathbf{x}'| - k|\mathbf{x}_2 - \mathbf{x}'|}}{|\mathbf{x}_1 - \mathbf{x}'| |\mathbf{x}_2 - \mathbf{x}'|} d^3x'. \quad (64)$$

Thus it is only the source intensity distribution that affects the field coherence. Comparison of this equation with Eq. (47) shows that this is a generalization of the van Cittert–Zernike theorem to primary sources. It is clear from this equation that the complex degree of spectral coherence depends only on the shape of the source and not the fact that the
source is completely incoherent. For a completely incoherent, planar, secondary source the radiant intensity is given by

\[ J_s(s) = J_0 \cos^2 \theta \]  

independent of the shape of the illuminated area on the source plane, where \( \theta \) is the angle that \( s \) makes with the normal to the source plane. This can be proven by substitution from Eq. (42) into Eq. (61). Note that such sources do not obey Lambert’s law. The far-field coherence, again, depends on the source intensity as given by the van Cittert–Zernike theorem [see Eq. (47)] and not on the source coherence.

**Spectrum**

For a quasi-homogeneous, three-dimensional primary source the spectrum of the radiation \( S^{0m}(R_s, \omega) \) at a point \( R_s \) in the direction \( s \) (unit vector) and at a distance \( R \) from the origin, in the far field of the source, can be found, as a function of the source spectrum \( S(0) \), by substitution from Eqs. (24) and (15) into Eq. (58) to get

\[ S^{0m}(R_s, \omega) = \frac{c^5}{\omega^3 R^3} \int \int \int \mu_0(\mathbf{x}', \omega) e^{-i k \mathbf{x}' \cdot s} d^3 \mathbf{x}' \]  

where we explicitly indicate the dependence of the complex degree of spectral coherence on frequency. Notice that the spectrum of the field is not necessarily the same as the spectrum of the source and, furthermore, that it can vary from point to point in space. The field spectrum depends on the source coherence as well as on the source spectrum. A very similar propagation relation can be found for the far-field spectrum from a planar, secondary source in the \( z = 0 \) plane. By substitution from Eqs. (24) and (15) into Eq. (61) we get

\[ S^{0m}(R_s, \omega) = \frac{S^{0m}(0)}{R^3} \int \int \mu_0(\mathbf{x}', \omega) e^{-i k \mathbf{x}' \cdot s} d^3 \mathbf{x}' \]

This shows that the spectrum for the field itself is changed upon propagation from the \( z = 0 \) plane into the far field and is different in different directions \( s \) from the source.

**4.7 SPECTRUM OF LIGHT**

**Limitations**

The complex analytic signal for a field that is not perfectly coherent, as defined in Eq. (3), is usually assumed to be a time-stationary random process. Therefore, the integral

\[ \int_{-\infty}^{\infty} u(x, t) e^{i\omega t} dt \]

does not converge, so that the analytic signal does not have a Fourier transform. Therefore, it is only possible to move freely from the space-time domain to the
space-frequency domain along the path shown in Fig. 5. Equation (3) does not apply to time-stationary fields within the framework of ordinary function theory.

Coherent Mode Representation

Coherent mode representation (see Wolf\(^{66,67}\)) has shown that any partially coherent field can be represented as the sum over component fields that are each perfectly self-coherent, but mutually incoherent with each other. Thus the cross-spectral density function for any field can be represented in the form

$$W_A(x_1, x_2) = \sum_n \lambda_n \phi_{A,n}(x_1) \phi_{A,n}^*(x_2)$$  \hspace{1cm} (69)

where \(\phi_{A,n}(x)\) is a phasor amplitude for its \(n\)th coherent component. This representation can be used either with primary \((A = Q)\), or secondary \((A = U)\) sources. The phasor amplitudes \(\phi_{A,n}(x)\) and the complex expansion coefficients \(\lambda_n\) in Eq. (69) are eigenfunctions and eigenvalues of \(W_A(x_1, x_2)\), as given by the equation

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_A(x_1, x_2) \phi_{A,n}(x_2) \, d^2 x_2 = \lambda_n(\omega) \phi_{A,n}(x_1)$$  \hspace{1cm} (70)

Since \(W_A(x_1, x_2)\) is hermitian, the eigenfunctions are complete and orthogonal, i.e.,

$$\sum_n \phi_{A,n}(x_1) \phi_{A,n}^*(x_2) = \delta^2(x_1 - x_2)$$  \hspace{1cm} (71)

and

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi_{A,n}(x_1) \phi_{A,m}^*(x_1) \, d^2 x_1 = \delta_{nm}$$  \hspace{1cm} (72)

where \(\delta_{nm}\) is the Kronecker delta function, and the eigenvalues are real and nonnegative definite, i.e.,

Real \(\{\lambda_n(\omega)\} \geq 0\)

Imag \(\{\lambda_n(\omega)\} = 0\)  \hspace{1cm} (73)
This is a very important concept in coherence theory. Wolf has used this representation to show that the frequency decomposition of the field can be defined in a different manner than was done in Eq. (3). A new phasor,

$$\tilde{U}_n(x, \omega) = \sum a_n(\omega) \phi_{x, n}(x, \omega)$$  \hspace{1cm} (74)

where $a_n(\omega)$ are random coefficients such that

$$\langle a_n(\omega) \bar{a}_m(\omega) \rangle = \lambda_n(\omega) \delta_{nm}$$  \hspace{1cm} (75)

can be introduced to represent the spectral components of the field. It then follows\textsuperscript{66} that, if the $\phi_{x, n}(x, \omega)$ are eigenfunctions of the cross-spectral density function, $W_{\phi}(x, x)$, then the cross-spectral density function can be represented as the correlation function between these phasors at the two spatial points $x$ and $x$, i.e.,

$$W_{\phi}(x_1, x_2) = \langle \tilde{U}_n(x_1, \omega) \bar{U}_m(x_2, \omega) \rangle$$  \hspace{1cm} (76)

in a manner very similar to the representation given in Eq. (11) in respect to the older phasors. Notice, by comparison of Eqs. (11) and (76), that the phasors $\tilde{U}_n(x, \omega)$ and $U_n(x, \omega)$ are not the same. One may formulate coherence theory either by defining $u(x, t)$ in Fig. 5 and then moving in a counterclockwise direction in this figure to derive the correlation functions using the Wiener-Khintchine theory or, alternatively, defining $\tilde{U}_n(x, \omega)$ and moving in a clockwise direction to define the correlation functions using the coherent mode expansion.

**Wolf Shift and Scaling law**

From Eqs. (66) and (67) it is clear that the spectrum of a radiation field may change as the field propagates. It is not necessarily equal to the source spectrum as it is usually assumed. This brings up an important question. Why do the spectra for light fields appear so constant, experimentally, that a change with propagation was never suspected? Wolf\textsuperscript{65} has provided at least part of the answer to this question. He discovered a scaling law that is obeyed by most natural fields and under which the normalized spectrums for most fields do remain invariant as they propagate. We can derive this scaling law by substitution from Eqs. (66) and (67) into Eq. (16). We then find that, if the complex degree of spectral coherence for the source is a function of $kx$. only, i.e.,

$$\mu_\langle x, \omega = f(kx)$$  \hspace{1cm} (77)

(so that this function is the same for each frequency component of the field, provided that the spatial separations of the two test points are always scaled by the wavelength), then the normalized spectrum in the far field is given by

$$s_\phi(R, \omega) = \frac{S_\phi(\omega)/\omega^3}{\int_0 S_\phi(\omega)/\omega^3 d\omega} \neq f(R)$$  \hspace{1cm} (78)
if $S_Q(\nu)$ is the complex degree of spectral coherence for a primary, quasi-homogeneous source [see Ref. 90, Eq. (65)], and

$$S_0^{v^*}(R_s, \omega) = \frac{S_0^{v^*}(\omega)}{\int S_0^{v^*}(\omega) d\omega} = f(R_s)$$  \hspace{1cm} (79)$$

if $S_0^{v^*}(\omega)$ is the normalized spectrum for a secondary, quasi-homogeneous source [see Ref. 90, Eq. (51)]. In each case the field spectrum does not change as the field propagates. Since the cross-spectral density function for a thermal source [see Eq. (43)] obeys this scaling law, it is not surprising that these changes in the spectrum of a propagating light field were never discovered experimentally. The fact that the spectrum can change was verified experimentally only after coherence theory pointed out the possibility.68–72

### 4.8 POLARIZATION EFFECTS

**Explicit Vector Representations**

As discussed earlier under "Scalar Field Amplitude", the scalar amplitude is frequently all that is required to treat the vector electromagnetic field. If polarization effects are important, it might be necessary to use two scalar field amplitudes to represent the two independent polarization components. However, in some complicated problems it is necessary to consider all six components of the vector field explicitly. For such a theory, the correlation functions between vector components of the field become tensors,73,74 which propagate in a manner very similar to the scalar correlation functions.

### 4.9 APPLICATIONS

**Speckle**

If coherent light is scattered from a stationary, rough surface, the phase of the light field is randomized in space. The diffraction patterns observed with such light displays a complicated granular pattern usually called speckle (see Ref. 8, sec. 7.5). Even though the light phase can be treated as a random variable, the light is still perfectly coherent. Coherence theory deals with the effects of time fluctuations, not spatial variations in the field amplitude or phase. Despite this, the same statistical tools used in coherence theory have been usefully applied to studying speckle phenomena.75,76 To treat speckle, the ensemble is usually redefined, not to represent the time fluctuations of the field but, rather, to represent all of the possible speckle patterns that might be observed under the conditions of a particular experiment. An observed speckle pattern is usually due to a single member of this ensemble (unless time fluctuations are also present), whereas the intensity observed in coherence theory is always the result of a weighted average over all of the ensemble. To obtain the intensity distribution over some plane, as defined in coherence theory, it would be necessary to average over all of the possible speckle patterns explicitly. If this is done, for example, by moving the scatterer while the intensity of a diffraction pattern is time-averaged, then time fluctuations are introduced into the field during the measurement; the light becomes partially coherent; and coherence theory can be properly used to model the measured intensity. One must be very careful in applying
the coherence theory model to treat speckle phenomena, because coherence theory was not originally formulated to deal with speckle.

**Statistical Radiometry**

Classical radiometry was originally based on a mechanical treatment of light as a flux of particles. It is not totally compatible with wave theories. Coherence theory has been used to incorporate classical radiometry into electromagnetic theory as much as has been found possible. It has been found that the usual definitions for the radiance function and the radiant emittance cause problems when applied to a wave theory. Some of this work is discussed earlier (pp. 4.8 to 4.10, 4.12, 4.13, 4.15 to 4.16, 4.18). Other radiometric functions, such as the radiant intensity, have clear meaning in a wave theory.

**Spectral Representation**

It was discovered, using coherence theory, that the spectrum of a light field is not the same as that of its source and that it can change as the light field propagates away from its source into the radiation field. Some of this work was discussed earlier under “Perfectly Incoherent Sources” and “Coherent Mode Representation.” This work has been found very useful for explaining troublesome experimental discrepancies in precise spectroradiometry.

**Laser Modes**

Coherence theory has been usefully applied to describing the coherence properties of laser modes. This theory is based on the coherent mode representation discussed under “Spectrum of Light.”

**Radio Astronomy**

Intensity interferometry was used to apply techniques from radio astronomy to observation with light. A lively debate ensued as to whether optical interference effects (which indicate partial coherence) could be observed from intensity correlations.

From relations like Eq. (26) and similar calculations using quantum coherence theory, it quickly became clear that they could. More recently, coherence theory has been used to model a radio telescope and to study how to focus an instrument to observe emitters that are not in the far field of the antenna array. It has been shown that a radio telescope and a conventional optical telescope are very similar, within a coherence theory model, even though their operation is completely different. This model makes the similarities between the two types of instruments very clear.

**Noncosmological Red Shift**

Cosmological theories for the structure and origin of the universe make great use of the observed red shift in the spectral lines of the light received from distant radiating objects, such as stars. It is usually assumed that the spectrum of the light is constant upon propagation and that the observed red shift is the result of simple Doppler shift due to the motion of the distant objects away from the earth in all directions. If this is true, then clearly the observed universe is expanding and must have begun with some sort of
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explosion, called “the big bang.” The size of the observable universe is estimated based on the amount of this red shift. A new theory by Wolf shows that red shifts can occur naturally without Doppler shifts as the light propagates from the source to an observer if the source is not in thermal equilibrium, i.e., a thermal source as discussed earlier under “Thermal (Lambertian) Source.” The basis of Wolf’s theory was discussed on pp. 4: 32–33 and 34–36.

4.10 REFERENCES


4.28 PHYSICAL OPTICS


