CHAPTER 9
GRADIENT INDEX OPTICS

Duncan T. Moore
The Institute of Optics
and
Gradient Lens Corporation
Rochester, New York

9.1 GLOSSARY

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>constant</td>
</tr>
<tr>
<td>a, b</td>
<td>constants</td>
</tr>
<tr>
<td>g</td>
<td>constant</td>
</tr>
<tr>
<td>h_i</td>
<td>constants</td>
</tr>
<tr>
<td>n</td>
<td>refractive index</td>
</tr>
<tr>
<td>r</td>
<td>radius</td>
</tr>
<tr>
<td>V_i</td>
<td>Abbe numbers</td>
</tr>
<tr>
<td>z</td>
<td>cartesian coordinate (optical axis direction)</td>
</tr>
<tr>
<td>Φ</td>
<td>power</td>
</tr>
</tbody>
</table>

9.2 INTRODUCTION

Gradient index (GRIN) optics refers to the field of optics in which light propagates along a curved path. This contrasts with normal homogeneous materials in which light propagates in a rectilinear fashion. Other terms that have been used to describe this field are inhomogeneous optics, index of refraction gradients, and distributed index of refraction. The most familiar example of a gradient index phenomenon is the mirage when a road appears to be wet on a hot summer day. This can be understood by the fact that the road is absorbing heat, thus slightly raising the temperature of the air relative to the temperature a few meters above the surface. By the gas law, the density decreases, and therefore the index of refraction decreases. Light entering this gradient medium follows a curved path. The ray path, as shown in Fig. 1, is such that the ray propagates downward towards the road and then gradually upwards to the observer’s eye. The observer sees two images. One is the normal image propagating through the homogeneous material and the second is an image that is inverted and appears below the road surface. Thus, the index of refraction gradient acts as a mirror by gradual light refraction rather than reflection.
9.2 OPTICAL ELEMENTS

![Figure 1](image)

**FIGURE 1** Light from point A emits or reflects in all directions. Light propagating several meters above the heated road travels in a straight line. Light passing through the lower index of refraction region near the road undergoes a bending. This light appears to have come from below the road.

9.3 ANALYTIC SOLUTIONS

Over the approximately 150 years that gradient index optics has been studied, a wealth of very interesting analytic solutions has been published. A classic example was published by James Clerk Maxwell in 1850. Maxwell showed through geometrical optics that the ray paths in a spherically symmetric material whose index of refraction is given by

\[ n(r) = \frac{a}{b^2 + r^2} \]  

(1)

are circles. The object and the image lie on the surface of the sphere but, otherwise, the imaging is perfect between the conjugate points on the sphere. The medium between the object and the image is continuous with no discreet surfaces. A century later, Luneburg modified the system to allow for discontinuities of the index of refraction. While these have not been implemented in ordinary optical systems, they have, however, been shown to be useful in integrated optics.

A final example of a numerical solution is that of a radial (cylindrical) gradient in which the index of refraction varies perpendicular to a line. In 1954, Fletcher showed that if the index of refraction is given by

\[ n(r) = n_o \text{sech}(ar^2) \]  

(2)

then the ray paths inside the material in the meridional plane are sinusoidal. Nearly fifty years earlier, Wood had shown experimentally that the paths appeared to be sinusoidal. This solution has several important commercial applications. It is the basis of the Selfoc lens used in arrays for facsimile and photocopying machines and in endoscopes used for medical applications.

9.4 MATHEMATICAL REPRESENTATION

Most of the gradient index profiles are represented by a polynomial expansion. While these expansions are not necessarily the most desirable from the gradient materials manufacturing standpoint, they are convenient for determining the aberrations of systems embodying GRIN materials. There are basically two major representations for gradient index materials. The first, used by the Nippon Sheet Glass, is used exclusively by representing radial gradient components. In this case, the index of refraction is written as a function of the radial coordinate \( r \)

\[ N(r) = N_0(1 - Ar^2/2 + h_1 r^4 + h_2 r^6 + \cdots) \]  

(3)
The second method of representing index of refraction profiles is a polynomial expansion in both the radial coordinate $r$ and the optical axis coordinate $z$. In this case, the representation is

$$ N(r, z) = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} N_{ij} r^i z^j $$

where the coefficients $N_{ij}$ are the coefficients of the index of refraction polynomial. A pure axial gradient (in the $z$ direction) has coefficients only in the form of $N_{ij}$ and those in the radial would be of the form of $N_{i0}$. These representations for the index of refraction polynomial have been the basis of the aberration theory which was first developed for gradient index materials with discreet surfaces by Sands. These coefficients are wavelength dependent and are typically defined at three wavelengths. A gradient dispersion is defined by using a general Abbe number

$$ V_{ij} = N_{ij, d}/(N_{ij, F} - N_{ij, C}) $$

except for $i$ and $j$ both equal to zero. In the case for $i = j = 0$, then the Abbe number becomes the standard form, namely

$$ V_{00} = (N_{00, d} - 1)/(N_{00, F} - N_{00, C}) $$

The subscripts $d$, $F$, and $C$ refer to the wavelengths 0.5876, 0.4861, and 0.6563 μm, respectively. Unlike the normal dispersion of glasses where $V_{00}$ is between 20 and 90, the $V_{ij}$ can have negative and positive values or can be infinite (implying that the gradient is the same as both the red and the blue portions of the spectrum).

### 9.5 AXIAL GRADIENT LENSES

When the index of refraction varies in the direction of the optical axis (the $z$ direction), the bending of the light within the material is very small. Thus, the main feature of an axial gradient is its ability to correct aberrations rather than to add power to the lens. Sands showed that the effect of an axial gradient on monochromatic aberrations is exactly equivalent to that of an aspheric surface. In fact, one could convert any aspherical surface to an axial gradient with a spherical surface and have the same image performance to the third-order approximation. There is, however, one very important difference between aspheric surfaces and axial gradients, i.e., the variation of the index of refraction profile with wavelength. Since an aspheric is the same for all wavelengths, its effect on spherochromatism is established once the aspheric has been determined. Further, an asphere has no effect on paraxial axial or lateral chromatic aberrations. This is not the case for axial gradients. Since the index of refraction profile varies with wavelength, it is possible to significantly modify the spherochromatism of the lens and, in the case where the gradient extends from the front to the back surface, to affect the paraxial chromatic aberrations. Depending upon the dispersion of the gradient index material, the spherochromatism can be increased or decreased independent of the monochromatic correction. The effect of an axial gradient on paraxial axial chromatic aberration is best understood by placing a surface perpendicular to the optical axis in the middle of a single lens dividing it into two parts. The gradient dispersion implies that the medium will have one dispersion at the front surface and a different dispersion at the second surface. Thus, if one were to design a material in which the dispersion of the front surface is 60 and at the rear surface is 40, then the combination of a positive (convex surface on the front) and negative lens (concave surface on the back), reduces the chromatic aberration. This can only be done if the lens is meniscus. In that case, the theoretical front lens is plano-convex while the back
9.4 OPTICAL ELEMENTS

The effective region of the gradient is in the "region of sag." One is plano-concave. If the negative element has the higher dispersion (lower $V$ numbers), then it is possible to chromatize the lens by a proper bending of the lens surfaces. This was first shown in the infrared part of the spectrum using a zinc sulfide-zinc selenide gradient material.

The simplest example of an axial gradient is the linear profile in which the index of refraction is written as

$$N(z) = N_{00} + N_{01}z$$  \(7\)

The coefficient $N_{01}$ is an additional degree of freedom which can be used to correct any of the third-order monochromatic aberrations except Petzval curvature of field. There are two ways to approach the design of these lenses. In the case where the index of refraction profile does not continue to the rear surface (see Fig. 2), a simple formula can be used to relate the amount of index change to the F-number of the lens surface if the third-order spherical aberration and coma are to be correct to zero, namely,

$$\Delta n = (0.0375/(N_{00} - 1)^2)/f^2$$  \(8\)

in this formula, the important parameters are the index of refraction of the base material, $N_{00}$, the change in index of refraction, $\Delta n$, from the polar tangent to the maximum sag point, and the F-number of the lens. One sees that if the F-number of the lens is doubled, then the amount of index change necessary to correct the spherical aberration and coma to zero increases by a factor of 4. Thus, while it is possible to correct the spherical aberration of the singlet operating at $F/4$ with an index change of only 0.0094,
that same lens operating at F/1 will require an index change of 0.15. In most lenses, one never corrects the spherical aberration of individual elements to zero, but corrects the total amount of spherical aberration of all lens elements to zero.

Axial gradients have been used in a number of lens designs. Most of the work in this field has occurred in photographic objectives. In these cases, they offer a slight advantage over aspherics because of the chromatic variation of the gradient.

### 9.6 RADIAL GRADIENTS

In the most generalized case for radial gradients (one in which all coefficients are nonzero), it is possible not only to use the gradient for aberration correction, but also to modify the focal length of the lens. Independent of which representation is used, the coefficient of the parabolic term [Eq. (2) or Eq. (3)] dictates the amount of power that is introduced by the radial gradient component. Assuming only a radial gradient component, Eq. (4) can be expanded as

$$N(r) = N_{00} + N_{01}r^2 + N_{02}r^4 + \cdots$$  \hspace{1cm} (9)

Equating the terms in Eq. (3) and Eq. (9) gives

$$N_{00} = N_0 \quad \text{and} \quad N_{10} = -N_0A/2$$  \hspace{1cm} (10)

In the most general form, the power $\phi$, due to the radial gradient component, is written as

$$\phi = -N_0A^{0.5}\sin(A^{0.5}t)$$  \hspace{1cm} (11)

From Eq. (11), the length of the material $t$ determines the focal length of the system. In fact, depending on the choice of length, the power can be positive, negative, or zero. See Fig. 3b. A convenient variation on this formula is to determine the length at which light

---

**FIGURE 3** Diagram of radial gradient: (a) wood lens; (b) a long radial gradient lens illustrating period ray path.
entering the material collimated will be focused on the rear surface. This length is called
the quarter pitch length of the rod and is given by

$$P_{1/4} = \left(\frac{\pi}{2}\right)(-N_{00}/(2N_{10}))^{0.5}$$  \hspace{1cm} (12)

The full period length of the rod is simply four times Eq. (12).
For the case where the focal length (the reciprocal of the power) is long compared to its
thickness, this can be approximated by the formula (see Fig. 3a)

$$\phi = -2N_{10}f$$  \hspace{1cm} (13)

This simplifying formula was derived by the entomologist Exner\textsuperscript{13} in 1889 while he was
analyzing insect eyes and found them to have radial gradient components. Since the
dispersion of a gradient material can be positive, negative, or infinity, the implication is
that the paraxial axial chromatic aberration can be negative, positive, or zero. This leads to
the possibility of an achromatized singlet with flat surfaces; or, by combining the dispersion
of the gradient with that of the homogeneous materials, to single element lenses with
curved surfaces that are color-corrected.

The radial gradient lens with flat surface is a very important example, both from a
theoretical and a commercial standpoint. Consider such a lens with an object of infinity
where the lens is thin relative to its focal length. As has already been shown, the value of
$N_{00}$ and the thickness determine the focal length of such a lens. According to third-order
aberration theory,\textsuperscript{8} the only other term that can influence the third-order monochromatic
aberrations is the coefficient $N_{20}$. This term can be used to correct any one of the
third-order aberrations except Petzval curvature of field. The coefficient $N_{20}$ is normally
used to correct the spherical aberration; however, once this choice is made, there are no
other degrees of freedom to reduce other aberrations such as coma. It can be shown that
the coma in such a single element lens is very large if the lens is used at infinite conjugates.
Of course, if such a lens is used at unit magnification in a system which is symmetric about
the aperture stop, the coma (as well as the distortion and paraxial lateral color) is zero. As
the length of the rod increases, the approximation for the focal length becomes inaccurate
and the more rigorous formula given by Eq. (11) is appropriate. However, the rules
governing the aberration correction remain the same. That is, the choice of the value of
$N_{20}$, or in the Nippon Sheet Glass representation, in $h_{5}$, coefficient, corrects the third-order
spherical aberration to zero. In Fletcher’s original paper, he showed that rays propagating
in a material whose index of refraction is given by Eq. (2) would focus light in the
meridional plane periodically with no aberration along the length of such a rod. If one
expands a hyperbolic secant in a polynomial expansion, one obtains

$$N_{20} = 5N_{00}^2/6N_{10}$$  \hspace{1cm} (14)

The implication is that if $N_{20}$ is chosen according to Eq. (14), then not only is the spherical
aberration corrected, but so is the tangential field (that is, the sum of three times the
astigmatism plus the field curvature). Rawson\textsuperscript{14} showed that a more appropriate value for
$N_{20}$ was $3N_{00}^2/2N_{10}$. This is a compromise for the correction of sagittal and tangential fields.

The second limiting case is to use these rods with arbitrary length but at unit
magnification. This has important commercial applications in photocopying and fax
machines, for couplers for single-mode fibers, and in relays used in endoscopes. In all of
these systems, the magnification is ±1 and thus there is no need to correct the coma, the
distortion, or the lateral color. Thus, the choice of $N_{20}$ can be used to either correct the
spherical aberration or to achieve a compromise between the tangential and sagittal fields.
In one of the most common applications, a series of lenses is assembled to form an
array (see Fig. 4). In this case, the magnification between the object and the image must be a +1 with an inverted image halfway through the gradient index rods. Light from an object point is imaged through multiple GRIN rods depending on the numerical aperture of each of the rods. The effective numerical aperture of the array is significantly higher than that of a single rod. Theoretically, a full two-dimensional array can be constructed to image an entire two-dimensional object. In practice, to reduce costs the object is scanned by moving the object across the fixed lens array with either a charged couple device or a transfer drum used to record the image.

**9.7 RADIAL GRADIENTS WITH CURVED SURFACES**

While the radial gradient with flat surfaces offers tremendous commercial applications today, it has limited applications because of the large amount of coma that is introduced unless the lens system is used at unit magnification. Thus, it is often desirable to introduce other degrees of freedom that may improve the imagery. The simplest way to do this is to make one or both of the end cases curved. The ability to chromatize such a lens is not lost so long as the power resulting from the curved surfaces and that of the radial gradient maintain the same ratio (but with opposite sign) as that of the Abbe number of homogeneous material and the Abbe number of \( N_{\infty} \). Thus, the lens shape can be determined to reduce the coma to zero and the value of the \( N_{\infty} \) coefficient is chosen to eliminate the spherical aberration. An example of a curved lens with a radial gradient was developed by Nippon Sheet Glass for a compact disc player. In that case, it is not necessary to achromatize the lens since the source is a monochromatic laser diode, but it was necessary to extend the field and reduce the amount of spherical aberration simultaneously. It is also often desirable to place part of the power on the curvature rather than using the gradient to refract all of the light. This reduces the magnitude of the index change and makes the lens easier to manufacture.

In a radial gradient material with curved surfaces, it is possible to eliminate four out of five monochromatic aberrations, and any four can be chosen. However, these lenses tend to be very sensitive to slight manufacturing errors, as they require a very delicate balance between the coefficients of the gradient profile and typically have very large amounts of higher-order aberrations.

**9.8 SHALLOW RADIAL GRADIENTS**

An interesting compromise between an axial gradient and a radial gradient with power is the shallow radial gradient (SRGRIN). In this type of gradient, there is no power generated by the gradient (i.e., \( N_{\infty} = 0 \)). Like the axial gradient, it has no effect on Petzval
curvature of field, but its aberration correction is significantly different than that of axial gradients. Sands showed that in the case of an axial gradient, the important parameter is the differential refraction of the ray at the surface which causes an additional surface contribution. In the shallow radial gradient there is no surface contribution, since the $N_{10}$ coefficient is zero. All of the aberration correction is from the transfer contribution through the material. The implication of this fact is that the thickness of the shallow radial gradient is very important and, in fact, the most important parameter is the product of the thickness and the $N_{20}$ coefficient. Thus, if only a small index change can be manufactured, the same amount of aberration correction can be achieved by increasing the thickness of the element. The other significant difference between this gradient and a normal radial gradient is the sign of the index change. In most lenses designed to date, the index of refraction of a conventional radial gradient should be lower at the periphery than it is at the center, thus creating a positive lens. However, in the shallow radial gradient, the index of refraction should be higher at the periphery than at the center. This has also normally been the case in the axial gradient in which the index of refraction should be higher at the polar tangent plane than at the maximum sag point. This has important implications for the manufacturing process. Furthermore, the amount of index change necessary for shallow gradient correction is usually very small compared to the amount of index change needed in a regular radial gradient.

### 9.9 MATERIALS

While several materials systems have been proposed for forming gradient index materials, gradients have only been made for commercial applications in glasses and polymers. However, research has been conducted in zinc selenide-zinc sulfide, and germanium-silicon for the infrared portion of the spectrum, and in fluoride materials for the ultraviolet. However, none of these have reached the stage, at this writing, which can be commercialized. For glasses, several processes have been proposed. The most common method of making gradient index materials is by the ion exchange process. In this case, a glass containing a single valence ion (such as sodium, lithium, or potassium) is placed in a molten salt bath at temperatures between 400 and 600°C. The molten salt bath contains a different ion than that in the glass. The ions from the salt diffuse into the glass and exchange for an ion of equal valence in the glass. The variation in composition leads to a variation in index of refraction. The variation in index of refraction occurs due to the change of polarizability between the two ions and the slight change in the density of the material. In some cases, these two phenomena can cancel one another, producing a composition variation, but no corresponding change in index of refraction. A model for predicting the index refraction change as well as the chromatic variation of the gradient has been developed. In this system, it is clear that the maximum index change is limited by the changes in the properties of single valence ions. While very large index changes have been made (approaching 0.27), these gradients suffer from large amounts of chromatic aberration. In axial gradients, a large amount of chromatic aberration is desirable, as it normally improves the spherochromatism. In the case of radial gradients, however, it creates large paraxial axial chromatic aberration which is normally not desirable.

The manufacturing method is quite simple. If one wishes to make axial gradients, a sheet of glass is placed in a molten salt bath. Typical times for diffusion are a few days for diffusion depths of 3 to 7 mm at temperatures around 500°C. The higher the temperature, the faster the diffusion; however, at high temperature the glass will begin to deform. Lower temperatures increase the diffusion times. For radial gradients, one simply starts with glass with cylindrical symmetry and places the rods inside an ion exchange bath. In order to form good parabolic profiles, it is necessary for the ions to diffuse through the center.
Two other methods have been proposed for making gradients in glass. In the first, the gradient is formed by leaching or by stuffing in a sol-gel formed glass. This system has only shown to be applicable to radial gradients. After the glass is formed by the sol-gel (solution gelatin process), the glass is in a porous state where one of the components can be dissolved out in an acid bath or molecules can be stuffed into the glass to form the index of refraction gradient. By the leaching method, gradients have been formed in either titanium or zirconium. Index changes of up to 0.03 have been formed by this method. Alternatively, the glass can be stuffed with ions such as lead. The lead precipitates on the walls of the porous material whereupon it is included in the glass during the sintering step. While it is possible to get much larger changes using the method based on lead, both of these techniques suffer from large amounts of chromatic aberration.

A new method shown to be very useful for axial gradients is based on the fusion of glass slabs. The index of refraction of each slab is slightly different than its adjacent slab. Very large index of refraction changes can be formed by this technique ($\Delta n = 0.4$). Further, these materials can be made in apertures up to 100 mm.

Two basic methods for manufacturing of polymers for gradient index have been demonstrated. In the first, an exchange of one monomer for a monomer in a partially polymerized material forms a profile in the same way as the ion exchange method. In the second, ultraviolet light is used to induce photocopolymerization to form an index of refraction in the material.

9.10 REFERENCES

1. For a source of over 100 articles on Gradient-Index (GRIN) Optics, the reader is referred to a series of special issues in *Applied Optics*, GRIN I (April 1, 1980), GRIN II (March 15, 1982), GRIN III (Feb. 1, 1983), GRIN IV (June 1, 1984), GRIN V (December 15, 1985), GRIN VI (October 1, 1986), GRIN VII (February 1, 1988), GRIN VIII (October 1, 1990 and December 1, 1990), and GRIN IX (September 1, 1992).


